Lalín *Res. Number Theory(2016)2:17*
DOI 10.1007/s40993-016-0047-5

Research in Number Theory

RESEARCH Open Access

A new method for obtaining polylogarithmic Mahler measure formulas

Matilde N. Lalín*

*Correspondence:
mlalin@dms.umontreal.ca
Département de mathématiques et de statistique, Université de Montréal., CP 6128, succ. Centre-ville, Montreal, QC H3C 3J7, Canada

Abstract

Given a formula for the Mahler measure of a rational function expressed in terms of polylogarithms, we describe a new method that allows us to construct a rational function with 2 more variables and whose Mahler measure is still expressed in terms of polylogarithms. We use this method to exhibit three new examples of Mahler measure and higher Mahler measure formulas. One of them involves a single term with $\zeta(5)$.

Keywords: Mahler measure, Higher Mahler measure, Special values of $\zeta(s)$ and Dirichlet L-functions, Polylogarithms

Mathematics Subject Classification: Primary 11R06; Secondary 11M06, 11R09

1 Background

Given a nonzero multivariable rational function $P \in \mathbb{C}(x_1, \dots, x_n)$, its (logarithmic) Mahler measure is defined by

$$\mathrm{m}(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $\mathbb{T}^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n | |z_1| = \cdots = |z_n| = 1\}$.

This construction originated in the search for large prime numbers [15] as well as in relation to classical heights of polynomials [16]. Later Mahler measure was found to yield special values of functions of number theoretic significance, such as the Riemann zeta function and other L-functions. Deninger explained the appearance of some L-functions in Mahler measure in terms of Beilinson's conjectures via relationships with regulators in [8]. (See also the works of Boyd [7] and Rodriguez-Villegas [17] for additional insight.) This point of view has shown that the Riemann zeta function and the L-functions that appear in Mahler measure formulas come from special values of polylogarithms arising from regulators.

The definition of Mahler measure can be generalized in the following way. For k a positive integer and the rest of the notation as before, the k-higher Mahler measure of P is given by

$$\mathrm{m}_k(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log^k |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

Higher Mahler measures were originally defined in [10] and subsequently studied by several authors. A related object, the Zeta Mahler measure, was first studied by Akatsuka in [1] who gave a first formula of a higher Mahler measure of a polynomial with a free coefficient in terms of polylogarithms. Deninger remarked that higher Mahler measures are expected to yield different regulators than the ones that appear in the case of the usual Mahler measure (see [13] for more details). Therefore, there is an interest in generating higher Mahler measure formulas involving polylogarithms that could potentially be related to regulators.

In [11,12] a method was introduced for increasing the number of variables in a Mahler measure formula involving polylogarithms. More precisely, we are given a rational function $P_a \in \mathbb{C}(x_1, \dots, x_m)$ such that its coefficients are rational functions in a parameter $a \in \mathbb{C}$. By making the change of variables $a = \left(\frac{1-z_1}{1+z_1}\right) \cdots \left(\frac{1-z_n}{1+z_n}\right)$, we can see the rational function P_a as a new rational function in $m+n$ variables, namely $\tilde{P} \in \mathbb{C}(x_1, \dots, x_m, z_1, \dots, z_n)$. If the Mahler measure of P_a has a formula involving multiple polylogarithms, it is possible, under favorable circumstances, to find a formula for the Mahler measure of $\tilde{P}$ also in terms of multiple polylogarithms. This construction also applies to higher Mahler measure [14]. By using this method with $P_a = x + a$, $P_a = (1+x)w + a(1+y)$, and $P_a = 1 + ax + (1-a)y$, formulas for

$$\mathrm{m}_k\left(x + \left(\frac{1-z_1}{1+z_1}\right) \cdots \left(\frac{1-z_n}{1+z_n}\right)\right),$$
$$\mathrm{m}\left((1+x)w + \left(\frac{1-z_1}{1+z_1}\right) \cdots \left(\frac{1-z_n}{1+z_n}\right)(1+y)\right),$$
$$\mathrm{m}\left(1 + \left(\frac{1-z_1}{1+z_1}\right) \cdots \left(\frac{1-z_n}{1+z_n}\right)x + \left(1 - \left(\frac{1-z_1}{1+z_1}\right) \cdots \left(\frac{1-z_n}{1+z_n}\right)\right)y\right),$$

for any integral $n \geq 1$ were established in [12,14].

The goal of this note is to introduce a different kind of change of variables, namely, $a = \frac{1+y}{1+z}$. This technique introduces two variables at the same time, and is considerably more involved.

As an application, we prove the following new formulas.

$$\mathrm{m}((1+z_1)(1+z_2) + (1+y_1)(1+y_2)x) = \frac{62}{\pi^4}\zeta(5), \tag{1}$$

$$\begin{aligned}
\mathrm{m}_2\left(1 + \left(\frac{1+y}{1+z}\right)x\right) &= \frac{37\pi^2}{360} + \frac{8}{\pi^2}\mathrm{Li}_{1,3}(1,-1) \\
&= \frac{16}{\pi^2}\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{14\log 2}{\pi^2}\zeta(3) \\
&\quad - \frac{23\pi^2}{360} - \frac{2\log^2 2}{3} + \frac{2\log^4 2}{3\pi^2}, \\
\mathrm{m}_3\left(1 + \left(\frac{1+y}{1+z}\right)x\right) &= -\frac{81}{4\pi^2}\zeta(5) + \frac{29}{8}\zeta(3) - \frac{48}{\pi^2}\mathrm{Li}_{1,1,3}(1,1,-1) \\
&= \frac{96}{\pi^2}\mathrm{Li}_5\left(\frac{1}{2}\right) - \frac{279}{4\pi^2}\zeta(5) + \frac{96\log 2}{\pi^2}\mathrm{Li}_4\left(\frac{1}{2}\right) \\
&\quad - \frac{3}{8}\zeta(3) + \frac{42\log^2 2}{\pi^2}\zeta(3) - \frac{8\log^3 2}{3} \\
&\quad + \frac{16\log^5 2}{5\pi^2},
\end{aligned} \tag{2}$$

where $\zeta(w)$ denotes the Riemann zeta function and $\mathrm{Li}_{n_1 \cdots n_m}(w_1, \dots, w_m)$ denotes the multiple polylogarithm, as defined later in Sect. 3, Eq. (3).

Formula (1) is particularly intriguing due to its simplicity and the appearance of the isolated term involving $\zeta(5)$. It is similar to the result

$$\mathrm{m}((1+w_1)(1+w_2)(1+z)+(1-w_1)(1-w_2)(1+y)x)=\frac{93}{\pi^4}\zeta(5)$$

proved in [11].

It should also be noted that examples of proven explicit evaluations of higher Mahler measure are hard to discover, specially for $k \geq 3$. We have reasons to believe that higher Mahler measure is more directly related to the regulator for k odd. Because of these considerations, Eq. (2), with its right-hand side with length-one polylogarithms, is very promising for the general theory.

The paper is organized as follows. In Sect. 2 we present our method that permits to construct all these new examples. We review the main definitions and properties of multiple polylogarithms in Sect. 3. We apply the method to recover known formulas and to prove new formulas in Sects. 4 and 5 respectively. Finally, we discuss future directions in Sect. 6.

2 A change of variables

Since higher Mahler measure includes classical Mahler measure when $k = 1$, we will often use the term "higher Mahler measure" to include both higher and classical Mahler measure.

Let $P_a \in \mathbb{C}(x_1,\dots,x_m)$ be a nonzero rational function such that its coefficients are rational functions in a parameter $a \in \mathbb{C}$. Further suppose that $\mathrm{m}_k(P_a)=\mathrm{m}_k(P_{|a|})$, that is, the higher Mahler measure only depends on the absolute value of the parameter a. Let $\tilde{P}=P_{\frac{1+y}{1+z}}(x_1,\dots,x_m)\in\mathbb{C}(x_1,\dots,x_m,y,z)$. We have

$$\begin{aligned}\mathrm{m}_k(\tilde{P}) &= \frac{1}{(2\pi i)^{m+2}}\int_{\mathbb{T}^{m+2}}\log^k|\tilde{P}|\frac{dx_1}{x_1}\cdots\frac{dx_m}{x_m}\frac{dy}{y}\frac{dz}{z}\\ &= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\left(\frac{1}{(2\pi i)^m}\int_{\mathbb{T}^m}\log^k|\tilde{P}|\frac{dx_1}{x_1}\cdots\frac{dx_m}{x_m}\right)\frac{dy}{y}\frac{dz}{z}\\ &= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\mathrm{m}_k\left(P_{\frac{1+y}{1+z}}\right)\frac{dy}{y}\frac{dz}{z}.\end{aligned}$$

The above formula shows that the higher Mahler measure of $\tilde{P}$ may be thought of as a double integral of the function that gives the higher Mahler measure of P_a.

Letting $y=-e^{4i\alpha}$ and $z=-e^{4i\beta}$, and keeping in mind that $\mathrm{m}_k(P_a(x_1,\dots,x_m)) = \mathrm{m}_k(P_{|a|}(x_1,\dots,x_m))$, the above becomes

$$\begin{aligned}&\frac{4}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\mathrm{m}_k\left(P_{\frac{\sin(\alpha)\cos(\alpha)}{\sin(\beta)\cos(\beta)}}\right)d\alpha\, d\beta\\ &\quad=\frac{4}{\pi^2}\int_{0\le\alpha\le\beta\le\frac{\pi}{2}}\mathrm{m}_k\left(P_{\frac{\sin(\alpha)\cos(\alpha)}{\sin(\beta)\cos(\beta)}}\right)d\alpha d\beta+\frac{4}{\pi^2}\int_{0\le\beta\le\alpha\le\frac{\pi}{2}}\mathrm{m}_k\left(P_{\frac{\sin(\alpha)\cos(\alpha)}{\sin(\beta)\cos(\beta)}}\right)d\alpha\, d\beta.\end{aligned}$$

We introduce the change of variables

$$s_1=\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\sin(\beta)},\qquad s_2=\frac{\sin(\alpha)\cos(\alpha)}{\sin(\beta)\cos(\beta)}.$$

The Jacobian has a surprisingly elegant formula:

$$\begin{aligned}
\begin{vmatrix} \frac{\partial s_1}{\partial \alpha} & \frac{\partial s_1}{\partial \beta} \\ \frac{\partial s_2}{\partial \alpha} & \frac{\partial s_2}{\partial \beta} \end{vmatrix} &= \begin{vmatrix} \frac{\cos(\beta)}{\cos^2(\alpha)\sin(\beta)} & -\frac{\sin(\alpha)}{\cos(\alpha)\sin^2(\beta)} \\ \frac{\cos^2(\alpha)-\sin^2(\alpha)}{\sin(\beta)\cos(\beta)} & -\frac{\sin(\alpha)\cos(\alpha)(\cos^2(\beta)-\sin^2(\beta))}{\sin^2(\beta)\cos^2(\beta)} \end{vmatrix} \\
&= -\frac{\sin(\alpha)(\cos^2(\beta)-\sin^2(\beta))}{\cos(\alpha)\cos(\beta)\sin^3(\beta)} + \frac{\sin(\alpha)(\cos^2(\alpha)-\sin^2(\alpha))}{\cos(\alpha)\cos(\beta)\sin^3(\beta)} \\
&= \frac{\sin(\alpha)}{\cos(\alpha)\cos(\beta)\sin^3(\beta)}(\sin^2(\beta)-\sin^2(\alpha)+\cos^2(\alpha)-\cos^2(\beta)) \\
&= 2\frac{\sin(\alpha)(\sin^2(\beta)-\sin^2(\alpha))}{\cos(\alpha)\cos(\beta)\sin^3(\beta)} \\
&= 2s_2(1-s_1^2).
\end{aligned}$$

In other words, we have that

$$d\alpha d\beta = \frac{ds_1 ds_2}{2s_2(1-s_1^2)}.$$

Notice that $0 \leq \alpha \leq \beta \leq \frac{\pi}{2}$ implies $0 \leq s_1 \leq s_2$ and $s_1 s_2 \leq 1$. From this we conclude $s_1 \leq \frac{1}{s_2} \leq \frac{1}{s_1}$ and $s_1 \leq 1$. Thus, we have two cases according to whether s_2 is less or greater than 1:

A. $0 \leq s_1 \leq s_2 \leq 1$ or
B. $0 \leq s_1 \leq \dfrac{1}{s_2} \leq 1$.

If, on the other hand, $0 \leq \beta \leq \alpha \leq \frac{\pi}{2}$, then $0 \leq s_2 \leq s_1$ and $s_1 s_2 \geq 1$. Thus, $\frac{1}{s_1} \leq \frac{1}{s_2} \leq s_1$ and $s_1 \geq 1$. Again, we obtain two cases depending on whether s_2 is less or greater than 1:

C. $0 \leq \dfrac{1}{s_1} \leq s_2 \leq 1$ or
D. $0 \leq \dfrac{1}{s_1} \leq \dfrac{1}{s_2} \leq 1$.

Thus we must integrate over $A \cup B \cup C \cup D$. Consider the changes of variables $t_i = \frac{1}{s_i}$. Thus, $\frac{ds_1}{1-s_1^2} = \frac{dt_1}{1-t_1^2}$ and $\frac{ds_2}{s_2} = -\frac{dt_2}{t_2}$. We apply these changes when necessary to make all the terms look similar. These changes of variables also revert the orientation of the integration domains in such a way that all the extra signs cancel. We obtain

$$\begin{aligned}
\mathrm{m}_k(\tilde{P}) &= \frac{2}{\pi^2}\int_{0\leq s_1\leq s_2\leq 1} \mathrm{m}_k\left(P_{s_2}\right)\frac{ds_1}{1-s_1^2}\frac{ds_2}{s_2} + \frac{2}{\pi^2}\int_{0\leq s_1\leq t_2\leq 1} \mathrm{m}_k\left(P_{1/t_2}\right)\frac{ds_1}{1-s_1^2}\frac{dt_2}{t_2} \\
&\quad + \frac{2}{\pi^2}\int_{0\leq t_1\leq s_2\leq 1} \mathrm{m}_k\left(P_{s_2}\right)\frac{dt_1}{1-t_1^2}\frac{ds_2}{s_2} + \frac{2}{\pi^2}\int_{0\leq t_1\leq t_2\leq 1} \mathrm{m}_k\left(P_{1/t_2}\right)\frac{dt_1}{1-t_1^2}\frac{dt_2}{t_2} \\
&= \frac{4}{\pi^2}\int_{0\leq s_1\leq s_2\leq 1} \mathrm{m}_k\left(P_{s_2}\right)\frac{ds_1}{1-s_1^2}\frac{ds_2}{s_2} + \frac{4}{\pi^2}\int_{0\leq s_1\leq s_2\leq 1} \mathrm{m}_k\left(P_{1/s_2}\right)\frac{ds_1}{1-s_1^2}\frac{ds_2}{s_2}.
\end{aligned}$$

Putting both terms together, we arrive at our main formula.

Theorem 1 *Let $P_a \in \mathbb{C}(x_1, \ldots, x_m)$ be a nonzero rational function such that its coefficients are rational functions in a parameter $a \in \mathbb{C}$ and such that $\mathrm{m}_k(P_a) = \mathrm{m}_k(P_{|a|})$. Then*

$$\mathrm{m}_k(\tilde{P}) = \frac{2}{\pi^2}\int_{0\leq s_1\leq s_2\leq 1}\left(\mathrm{m}_k\left(P_{s_2}\right) + \mathrm{m}_k\left(P_{1/s_2}\right)\right)\left(\frac{1}{s_1+1} - \frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2},$$

where $\tilde{P} = P_{\frac{1+y}{1+z}}(x_1, \ldots, x_m) \in \mathbb{C}(x_1, \ldots, x_m, y, z)$.

3 Polylogarithms and hyperlogarithms

We will apply Theorem 1 when $\mathrm{m}_k(P_a)$ is given by multiple polylogarithms. Here we recall the main definitions and tools that we will need from this topic.

Let $w_1, \ldots, w_m$ be complex variables and $n_1, \ldots, n_m$ be positive integers. Define the multiple polylogarithm by the power series

$$\mathrm{Li}_{n_1 \cdots n_m}(w_1, \ldots, w_m) := \sum_{0<j_1<\cdots<j_m} \frac{w_1^{j_1} \cdots w_m^{j_m}}{j_1^{n_1} \cdots j_m^{n_m}}. \tag{3}$$

We say that the above series has length m and weight $\omega = n_1 + \cdots + n_m$. It is absolutely convergent for $|w_i| \leq 1$ and $n_m > 1$.

Specializing the $w_i = 1$ we recover multizeta values. In particular, the Riemann zeta function arises as the polylogarithm of length one specialized at $w_1 = 1$.

Multiple polylogarithms have meromorphic continuations to the complex plane. Hyperlogarithms are defined by the following iterated integral.

$$\mathrm{I}_{n_1 \cdots n_m}(a_1 : \cdots : a_{m+1}) := \int_0^{a_{m+1}} \underbrace{\frac{dt}{t-a_1} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n_1} \circ \underbrace{\frac{dt}{t-a_2} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n_2}$$
$$\circ \cdots \circ \underbrace{\frac{dt}{t-a_m} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n_m}$$

where

$$\int_0^{b_{h+1}} \frac{dt}{t-b_1} \circ \cdots \circ \frac{dt}{t-b_h} = \int_{0 \leq t_1 \leq \cdots \leq t_h \leq b_{h+1}} \frac{dt_1}{t_1 - b_1} \cdots \frac{dt_h}{t_h - b_h}.$$

The path of integration should be interpreted as any path connecting 0 and b_{h+1} in $\mathbb{C} \setminus \{b_1, \ldots, b_h\}$. The integral depends on the homotopy class of this path. For our purposes, we will always integrate in the real line.

Multiple polylogarithms and hyperlogarithms are related by the following identities (see [9]).

$$\mathrm{I}_{n_1 \cdots n_m}(a_1 : \cdots : a_{m+1}) = (-1)^m \mathrm{Li}_{n_1 \cdots n_m}\left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots, \frac{a_{m+1}}{a_m}\right),$$
$$\mathrm{Li}_{n_1 \cdots n_m}(w_1, \ldots, w_m) = (-1)^m \mathrm{I}_{n_1 \cdots n_m}((w_1 \cdots w_m)^{-1} : \cdots : w_m^{-1} : 1).$$

There are relations over $\mathbb{Q}$ among polylogarithms. We record here a useful result for later.

Theorem 2 (Borwein and Girgensohn, (75) in [4]) *For $\rho, \sigma = \pm 1$ and $r + s$ odd, we have*

$$\mathrm{Li}_{r,s}(\rho, \sigma) = \frac{1}{2}(-\mathrm{Li}_{r+s}(\rho\sigma) + (1 + (-1)^s)\mathrm{Li}_r(\rho)\mathrm{Li}_s(\sigma))$$
$$+ \frac{(-1)^s}{2}\left(\binom{r+s-1}{r-1}\mathrm{Li}_{r+s}(\rho) + \binom{r+s-1}{s-1}\mathrm{Li}_{r+s}(\sigma)\right)$$

$$-(-1)^s \sum_{0<k<\frac{r+s}{2}} \mathrm{Li}_{2k}(\rho\sigma)\left(\binom{r+s-2k-1}{r-1}\mathrm{Li}_{r+s-2k}(\rho)\right.$$
$$\left.+\binom{r+s-2k-1}{s-1}\mathrm{Li}_{r+s-2k}(\sigma)\right).$$

When $\mathrm{m}_k\left(P_a\right)$ and $\mathrm{m}_k\left(P_{1/a}\right)$ are related to multiple polylogarithms, and therefore to hyperlogarithms, the integral in Theorem 1 may also be related to hyperlogarithms, and this yields a formula relating $\mathrm{m}_k(\tilde{P})$ to multiple polylogarithms.

4 Some examples

In this section we apply Theorem 1 to a couple of well-known examples. While we do not recover any new results here, we include this section because it allows the reader to appreciate how the method works.

First consider the case $P_a(x) = x + a$ and $k = 1$. Then Jensen's formula implies that

$$\mathrm{m}(x+a) = \log^+|a| := \begin{cases} 0 & \text{for } |a| \le 1, \\ \log|a| & \text{for } |a| \ge 1. \end{cases}$$

By applying Theorem 1, we obtain

$$\mathrm{m}(\tilde{P}) = \mathrm{m}\left(x + \frac{1+y}{1+z}\right) = \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(\frac{1}{s_1+1} - \frac{1}{s_1-1}\right) ds_1 \frac{\log(1/s_2)ds_2}{s_2}.$$

If we write the logarithm in terms of the hyperlogarithm $\log s_2 = -\int_{s_2}^1 \frac{ds_3}{s_3}$, we get

$$\begin{aligned}\mathrm{m}(\tilde{P}) &= \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le s_3\le 1}\left(\frac{1}{s_1+1} - \frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}\frac{ds_3}{s_3}\\ &= \frac{2}{\pi^2}(I_3(-1:1) - I_3(1:1))\\ &= \frac{2}{\pi^2}(\mathrm{Li}_3(1) - \mathrm{Li}_3(-1))\\ &= \frac{7}{2\pi^2}\zeta(3).\end{aligned} \tag{4}$$

Since $\mathrm{m}(1+y) = 0$ and classical Mahler measure is multiplicative, we have $\mathrm{m}\left(x + \frac{1+y}{1+z}\right) = \mathrm{m}\left((1+z) + (1+y)x\right)$. Thus, we have recovered a result of Smyth (Appendix 1 in [6]).

We can go further and introduce a constant in the final result by writing $P_a = x + ab$ so that

$$\begin{aligned}\mathrm{m}(\tilde{P}) &= \mathrm{m}\left(x + b\left(\frac{1+y}{1+z}\right)\right)\\ &= \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(\frac{1}{s_1+1} - \frac{1}{s_1-1}\right) ds_1 \frac{(\log^+(|b|s_2) + \log^+(|b|/s_2))ds_2}{s_2}.\end{aligned}$$

Suppose first that $|b| \le 1$. Then $|b|s_2 \le 1$ and

$$\begin{aligned}\mathrm{m}(\tilde{P}) &= \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(\frac{1}{s_1+1} - \frac{1}{s_1-1}\right) ds_1 \frac{\log^+(|b|/s_2)ds_2}{s_2}\\ &= \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le |b|}\left(\frac{1}{s_1+1} - \frac{1}{s_1-1}\right) ds_1 \frac{\log(|b|/s_2)ds_2}{s_2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le s_3\le |b|}\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right)ds_1\frac{ds_2}{s_2}\frac{ds_3}{s_3}\\
&= \frac{2}{\pi^2}(I_3(-1:|b|)-I_3(1:|b|))\\
&= \frac{2}{\pi^2}(\mathrm{Li}_3(|b|)-\mathrm{Li}_3(-|b|)).
\end{aligned}
$$

Now suppose that $|b|\ge 1$. Then $|b|/s_2\ge 1$ and

$$
\begin{aligned}
\mathrm{m}(\tilde{P}) &= \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right)ds_1\frac{(\log^+(|b|s_2)+\log(|b|/s_2))ds_2}{s_2}\\
&= \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right)ds_1\frac{(\log(|b|s_2)+\log(|b|/s_2))ds_2}{s_2}\\
&\quad -\frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1/|b|}\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right)ds_1\frac{\log(|b|s_2)ds_2}{s_2}\\
&= \frac{4}{\pi^2}\log|b|\int_{0\le s_1\le s_2\le 1}\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right)ds_1\frac{ds_2}{s_2}\\
&\quad +\frac{2}{\pi^2}\int_{0\le s_1\le s_2\le s_3\le 1/|b|}\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right)ds_1\frac{ds_2}{s_2}\frac{ds_3}{s_3}\\
&= \frac{4}{\pi^2}\log|b|(I_2(-1:1)-I_2(1:1))+\frac{2}{\pi^2}(I_3(-1:1/|b|)-I_3(1:1/|b|))\\
&= \frac{4}{\pi^2}\log|b|(\mathrm{Li}_2(1)-\mathrm{Li}_2(-1))+\frac{2}{\pi^2}(\mathrm{Li}_3(|b|^{-1})-\mathrm{Li}_3(-|b|^{-1}))\\
&= \log|b|+\frac{2}{\pi^2}(\mathrm{Li}_3(|b|^{-1})-\mathrm{Li}_3(-|b|^{-1})).
\end{aligned}
$$

In this way we have recovered another result of Smyth [18]. More precisely, for $b\in\mathbb{C}$ nonzero,

$$
\mathrm{m}((1+z)+b(1+y)x)=\begin{cases}\frac{2}{\pi^2}(\mathrm{Li}_3(|b|)-\mathrm{Li}_3(-|b|)) & \text{for } |b|\le 1,\\ \log|b|+\frac{2}{\pi^2}\left(\mathrm{Li}_3\left(|b|^{-1}\right)-\mathrm{Li}_3\left(-|b|^{-1}\right)\right) & \text{for } |b|>1.\end{cases} \tag{5}
$$

5 Main results

After gaining more understanding of how Theorem 1 can be applied to create new examples of Mahler measure, we proceed to prove new results.

Theorem 3 *We have the following formula involving a five-variable polynomial:*

$$
\mathrm{m}((1+z_1)(1+z_2)+(1+y_1)(1+y_2)x)=\frac{62}{\pi^4}\zeta(5).
$$

Proof Our goal is to integrate Formula (5) into Theorem 1. First, we need to write the difference of trilogarithms in terms of a convenient difference of hyperlogarithms. We have, for $0<a\le 1$

$$
\mathrm{Li}_3(a)-\mathrm{Li}_3(-a)=\int_0^1\left(\frac{1}{t+a^{-1}}-\frac{1}{t-a^{-1}}\right)dt\circ\frac{dt}{t}\circ\frac{dt}{t}.
$$

If we make the change of variables $s=ta$, this yields

$$
\mathrm{Li}_3(a)-\mathrm{Li}_3(-a)=\int_0^a\left(\frac{1}{s+1}-\frac{1}{s-1}\right)ds\circ\frac{ds}{s}\circ\frac{ds}{s}. \tag{6}
$$

Now we take $P_a = (1+z_1) + a(1+y_1)x$. Therefore $\tilde{P} = (1+z_1) + \frac{(1+y_2)}{(1+z_2)}(1+y_1)x$. Because $\mathrm{m}(1+z_2) = 0$, we have that $\mathrm{m}(\tilde{P}) = \mathrm{m}((1+z_1)(1+z_2) + (1+y_1)(1+y_2)x)$.

Theorem 1 implies

$$\mathrm{m}(\tilde{P}) = \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1} \left(\mathrm{m}\left(P_{s_2}\right) + \mathrm{m}\left(P_{1/s_2}\right)\right)\left(\frac{1}{s_1+1} - \frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}.$$

By Eq. (5),

$$\begin{aligned}
\mathrm{m}(\tilde{P}) = {} & \frac{4}{\pi^4}\int_{0\le s_1\le s_2\le 1} (\mathrm{Li}_3(s_2) - \mathrm{Li}_3(-s_2))\left(\frac{1}{s_1+1} - \frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}\\
& + \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(\log(1/s_2) + \frac{2}{\pi^2}(\mathrm{Li}_3(s_2) - \mathrm{Li}_3(-s_2))\right)\\
& \times \left(\frac{1}{s_1+1} - \frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}.
\end{aligned}$$

Recalling (4) and (6), and considering all the possible orderings (shuffles) that the variables can take inside the multidimensional integral,

$$\begin{aligned}
\mathrm{m}(\tilde{P}) = {} & \frac{7}{2\pi^2}\zeta(3) + \frac{8}{\pi^4}\int_{0\le s_1\le s_2\le 1}\left(\frac{1}{s_1+1} - \frac{1}{s_1-1}\right) ds_1\\
& \times \left(\int_0^{s_2}\left(\frac{1}{s+1} - \frac{1}{s-1}\right) ds \circ \frac{ds}{s} \circ \frac{ds}{s}\right)\frac{ds_2}{s_2}\\
= {} & \frac{7}{2\pi^2}\zeta(3) + \frac{16}{\pi^4}\int_0^1\left(\frac{1}{s+1} - \frac{1}{s-1}\right) ds \circ \left(\frac{1}{s+1} - \frac{1}{s-1}\right) ds \circ \frac{ds}{s} \circ \frac{ds}{s} \circ \frac{ds}{s}\\
& + \frac{8}{\pi^4}\int_0^1\left(\frac{1}{s+1} - \frac{1}{s-1}\right) ds \circ \frac{ds}{s}\left(\frac{1}{s+1} - \frac{1}{s-1}\right) ds \circ \frac{ds}{s} \circ \frac{ds}{s}\\
& + \frac{8}{\pi^4}\int_0^1\left(\frac{1}{s+1} - \frac{1}{s-1}\right) ds \circ \frac{ds}{s} \circ \frac{ds}{s}\left(\frac{1}{s+1} - \frac{1}{s-1}\right) ds \circ \frac{ds}{s}.
\end{aligned}$$

We write everything in terms of hyperlogarithms and polylogarithms,

$$\begin{aligned}
\mathrm{m}(\tilde{P}) = {} & \frac{7}{2\pi^2}\zeta(3) + \frac{16}{\pi^4}(I_{1,4}(-1:-1:1) - I_{1,4}(-1:1:1)\\
& + I_{1,4}(1:1:1) - I_{1,4}(1:-1:1))\\
& + \frac{8}{\pi^4}(I_{2,3}(-1:-1:1) - I_{2,3}(-1:1:1) + I_{2,3}(1:1:1) - I_{2,3}(1:-1:1))\\
& + \frac{8}{\pi^4}(I_{3,2}(-1:-1:1) - I_{3,2}(-1:1:1) + I_{2,3}(1:1:1) - I_{3,2}(1:-1:1))\\
= {} & \frac{7}{2\pi^2}\zeta(3) + \frac{16}{\pi^4}(\mathrm{Li}_{1,4}(1,-1) - \mathrm{Li}_{1,4}(-1,1) + \mathrm{Li}_{1,4}(1,1) - \mathrm{Li}_{1,4}(-1,-1))\\
& + \frac{8}{\pi^4}(\mathrm{Li}_{2,3}(1,-1) - \mathrm{Li}_{2,3}(-1,1) + \mathrm{Li}_{2,3}(1,1) - \mathrm{Li}_{2,3}(-1,-1))\\
& + \frac{8}{\pi^4}(\mathrm{Li}_{3,2}(1,-1) - \mathrm{Li}_{3,2}(-1,1) + \mathrm{Li}_{3,2}(1,1) - \mathrm{Li}_{3,2}(-1,-1)).
\end{aligned}$$

The terms in the above computation may be reduced using Theorem 2.

$$\mathrm{Li}_{1,4}(1,-1)-\mathrm{Li}_{1,4}(-1,1)+\mathrm{Li}_{1,4}(1,1)-\mathrm{Li}_{1,4}(-1,-1)=\frac{31}{16}\zeta(5)-\frac{7\pi^2}{48}\zeta(3),$$
$$\mathrm{Li}_{2,3}(1,-1)-\mathrm{Li}_{2,3}(-1,1)+\mathrm{Li}_{2,3}(1,1)-\mathrm{Li}_{2,3}(-1,-1)=-\frac{31}{4}\zeta(5)+\frac{35\pi^2}{48}\zeta(3),$$
$$\mathrm{Li}_{3,2}(1,-1)-\mathrm{Li}_{3,2}(-1,1)+\mathrm{Li}_{3,2}(1,1)-\mathrm{Li}_{3,2}(-1,-1)=\frac{93}{8}\zeta(5)-\frac{7\pi^2}{8}\zeta(3).$$

After all the simplifications, we get

$$\mathrm{m}(\tilde{P})=\frac{62}{\pi^4}\zeta(5),$$

and conclude in this way with the proof of Theorem 3. □

Before proceeding to our next result, we recall a theorem of higher Mahler measure due to Akatsuka.

Theorem 4 (Theorem 7, [1], Theorem 4.5, [14]) *Let*

$$L_{(n_1,\ldots,n_m)}(w):=\mathrm{Li}_{n_1,\ldots,n_m}(1,\ldots,1,w)=\sum_{0<j_1<\cdots<j_m}\frac{w^{j_m}}{j_1^{n_1}\cdots j_m^{n_m}}.$$

We have, for $|a|\leq 1$ *and* $k\geq 2$,

$$\mathrm{m}_k(x+a)=(-1)^k k!\sum_{\frac{k}{2}-1\leq n\leq k-2}\frac{1}{2^{2(k-n-1)}}\sum_{\substack{(\epsilon_1,\ldots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-n-2}}L_{(\epsilon_1,\ldots,\epsilon_n,2)}(|a|^2).$$

For $|a|\geq 1$ *and* $k\geq 2$,

$$\mathrm{m}_k(x+a)=\log^k|a|+\sum_{j=0}^{k-2}\binom{k}{j}(-1)^{k-j}(k-j)!\sum_{\frac{k-j}{2}-1\leq n\leq k-j-2}\frac{1}{2^{2(k-j-n-1)}}$$
$$\times\sum_{\substack{(\epsilon_1,\ldots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-j-n-2}}\log^j|a|L_{(\epsilon_1,\ldots,\epsilon_n,2)}(|a|^{-2}).$$

The above statement implies in particular that

$$\mathrm{m}_2(x+a)=\begin{cases}\frac{1}{2}\mathrm{Li}_2\left(|a|^2\right) & \text{for } |a|\leq 1,\\ \log^2|a|+\frac{1}{2}\mathrm{Li}_2\left(|a|^{-2}\right) & \text{for } |a|>1,\end{cases}\tag{7}$$

as well as

$$\mathrm{m}_3(x+a)=\begin{cases}-\frac{3}{2}L_{(1,2)}\left(|a|^2\right) & \text{for } |a|\leq 1,\\ \log^3|a|+\frac{3}{2}\log|a|\mathrm{Li}_2\left(|a|^{-2}\right)-\frac{3}{2}L_{(1,2)}(|a|^{-2}) & \text{for } |a|>1.\end{cases}\tag{8}$$

Theorem 5 *We have the following 2-higher Mahler measure evaluation*

$$\begin{aligned}\mathrm{m}_2\left(1+\left(\frac{1+y}{1+z}\right)x\right) &= \frac{37\pi^2}{360}+\frac{8}{\pi^2}\mathrm{Li}_{1,3}(1,-1)\\ &= \frac{16}{\pi^2}\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{14\log 2}{\pi^2}\zeta(3)-\frac{23\pi^2}{360}-\frac{2\log^2 2}{3}+\frac{2\log^4 2}{3\pi^2}.\end{aligned}$$

Proof Our goal is to integrate (7) into Theorem 1. By using hyperlogarithms, we have, for $0 < a \le 1$,

$$\mathrm{Li}_2\left(a^2\right) = -\int_0^1 \frac{dt}{t-a^{-2}} \circ \frac{dt}{t}.$$

If we set $s^2 = a^2 t$, we obtain $2sds = a^2 dt$ and

$$\mathrm{Li}_2\left(a^2\right) = -\int_0^a \frac{2sds}{s^2-1} \circ \frac{2ds}{s} = -2\int_0^a \left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \frac{ds}{s}. \tag{9}$$

Now we take $P_a = 1 + ax$. Therefore $\tilde{P} = 1 + \left(\frac{1+y}{1+z}\right)x$. Theorem 1 allows us to write

$$\mathrm{m}_2(\tilde{P}) = \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1} \left(\mathrm{m}_2\left(P_{s_2}\right)+\mathrm{m}_2\left(P_{1/s_2}\right)\right)\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}.$$

By Eq. (7),

$$\begin{aligned}\mathrm{m}_2(\tilde{P}) &= \frac{1}{\pi^2}\int_{0\le s_1\le s_2\le 1} \mathrm{Li}_2\left(s_2^2\right)\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}\\ &\quad + \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(\log^2(1/s_2)+\frac{1}{2}\mathrm{Li}_2\left(s_2^2\right)\right)\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}.\end{aligned}$$

We use Eq. (9) and consider all the possible shuffles of the variables inside the integral. This yields:

$$\begin{aligned}\mathrm{m}_2(\tilde{P}) &= \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(-2\int_0^{s_2}\left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \frac{ds}{s}\right)\\ &\quad \times \left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}\\ &\quad + \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}\left(\int_{s_2}^1 \frac{ds_3}{s_3}\right)^2\\ &= -\frac{4}{\pi^2}\int_0^1 \left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \frac{ds}{s} \circ \left(\frac{1}{s+1}-\frac{1}{s-1}\right) ds \circ \frac{ds}{s}\\ &\quad - \frac{4}{\pi^2}\int_0^1 \left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \left(\frac{1}{s+1}-\frac{1}{s-1}\right) ds \circ \frac{ds}{s} \circ \frac{ds}{s}\\ &\quad - \frac{4}{\pi^2}\int_0^1 \left(\frac{1}{s+1}-\frac{1}{s-1}\right) ds \circ \left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \frac{ds}{s} \circ \frac{ds}{s}\\ &\quad + \frac{4}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}\frac{ds_3}{s_3}\frac{ds_4}{s_4}.\end{aligned}$$

We write everything in terms of hyperlogarithms and polylogarithms:

$$\begin{aligned}
\mathrm{m}_2(\tilde{P}) &= \frac{4}{\pi^2}(-I_{2,2}(-1:-1:1)+I_{2,2}(-1:1:1)-I_{2,2}(1:-1:1)+I_{2,2}(1:1:1))\\
&\quad+\frac{8}{\pi^2}(-I_{1,3}(-1:-1:1)+I_{1,3}(1:1:1))+\frac{4}{\pi^2}(I_4(-1:1)-I_4(1:1)\\
&=\frac{4}{\pi^2}(-\mathrm{Li}_{2,2}(1,-1)+\mathrm{Li}_{2,2}(-1,1)-\mathrm{Li}_{2,2}(-1,-1)+\mathrm{Li}_{2,2}(1,1))\\
&\quad+\frac{8}{\pi^2}(-\mathrm{Li}_{1,3}(1,-1)+\mathrm{Li}_{1,3}(1,1))+\frac{4}{\pi^2}(\mathrm{Li}_4(1)-\mathrm{Li}_4(-1)).
\end{aligned}$$

By [2], we have the following reductions

$$\begin{aligned}
&-\mathrm{Li}_{2,2}(1,-1)+\mathrm{Li}_{2,2}(-1,1)-\mathrm{Li}_{2,2}(-1,-1)+\mathrm{Li}_{2,2}(1,1)\\
&\quad=4\mathrm{Li}_{1,3}(1,-1)-\frac{\pi^4}{1440},
\end{aligned}$$

and

$$\mathrm{Li}_{1,3}(1,1)=\frac{\pi^4}{360}.$$

We also have the well-known and easy to prove result

$$\mathrm{Li}_4(1)-\mathrm{Li}_4(-1)=\frac{\pi^4}{48}.$$

Combining all of the above identities, we reach a very simplified version of the formula.

$$\mathrm{m}_2(\tilde{P})=\frac{37\pi^2}{360}+\frac{8}{\pi^2}\mathrm{Li}_{1,3}(1,-1).$$

In order to express the above formula using polylogarithms of length one, we use this result from [5]

$$\mathrm{Li}_{1,3}(1,-1)=2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{7\log 2}{4}\zeta(3)-\frac{\pi^4}{48}-\frac{\log^2 2}{12}\pi^2+\frac{\log^4 2}{12},$$

and we finally obtain

$$\mathrm{m}_2(\tilde{P})=\frac{16}{\pi^2}\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{14\log 2}{\pi^2}\zeta(3)-\frac{23\pi^2}{360}-\frac{2\log^2 2}{3}+\frac{2\log^4 2}{3\pi^2}.$$

□

Theorem 6 *We have the following 3-higher Mahler measure evaluation*

$$\begin{aligned}
\mathrm{m}_3\left(1+\left(\frac{1+y}{1+z}\right)x\right)&=-\frac{81}{4\pi^2}\zeta(5)+\frac{29}{8}\zeta(3)-\frac{48}{\pi^2}\mathrm{Li}_{1,1,3}(1,1,-1)\\
&=\frac{96}{\pi^2}\mathrm{Li}_5\left(\frac{1}{2}\right)-\frac{279}{4\pi^2}\zeta(5)+\frac{96\log 2}{\pi^2}\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{3}{8}\zeta(3)\\
&\quad+\frac{42\log^2 2}{\pi^2}\zeta(3)-\frac{8\log^3 2}{3}+\frac{16\log^5 2}{5\pi^2}.
\end{aligned}$$

Proof Once again, we seek to integrate (8) into Theorem 1. We have

$$L_{(1,2)}\left(a^2\right)=\mathrm{Li}_{1,2}(1,a^2)=I_{1,2}(a^{-2}:a^{-2}:1)=\int_0^1\frac{dt}{t-a^{-2}}\circ\frac{dt}{t-a^{-2}}\circ\frac{dt}{t}.$$

After setting $s^2 = a^2 t$, we obtain

$$\begin{aligned} L_{(1,2)}\left(a^2\right) &= \int_0^a \frac{2sds}{s^2-1} \circ \frac{2sds}{s^2-1} \circ \frac{2ds}{s} \\ &= 2\int_0^a \left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \frac{ds}{s}. \end{aligned} \tag{10}$$

Thus, we use Theorem 1 once again in order to write

$$\mathrm{m}_3(\tilde{P}) = \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1} \left(\mathrm{m}_3\left(P_{s_2}\right)+\mathrm{m}_3\left(P_{1/s_2}\right)\right)\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}.$$

Now (8) implies

$$\begin{aligned} \mathrm{m}_3(\tilde{P}) &= \frac{1}{\pi^2}\int_{0\le s_1\le s_2\le 1} \left(-3L_{(1,2)}\left(s_2^2\right)\right)\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2} \\ &\quad + \frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1} \left(\log^3(1/s_2)+\frac{3}{2}\log(1/s_2)\mathrm{Li}_2\left(s_2^2\right)-\frac{3}{2}L_{(1,2)}\left(s_2^2\right)\right) \\ &\quad \times \left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}. \end{aligned}$$

By using Eqs. (9) and (10) the above equals

$$\begin{aligned} &-\frac{12}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(\int_0^{s_2}\left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \frac{ds}{s}\right) \\ &\quad \times \left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2} \\ &\quad -\frac{6}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(\int_0^{s_2}\left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \frac{ds}{s}\right) \\ &\quad \times \left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}\int_{s_2}^1 \frac{ds_3}{s_3} \\ &\quad +\frac{2}{\pi^2}\int_{0\le s_1\le s_2\le 1}\left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right) ds_1 \frac{ds_2}{s_2}\left(\int_{s_2}^1 \frac{ds_3}{s_3}\right)^3. \end{aligned}$$

Thus, by considering all the possible shuffles,

$$\begin{aligned} \mathrm{m}_3(\tilde{P}) &= -\frac{12}{\pi^2}\int_0^1 \left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \frac{ds}{s} \\ &\quad \circ \left(\frac{1}{s+1}-\frac{1}{s-1}\right) ds \circ \frac{ds}{s} \\ &\quad -\frac{12}{\pi^2}\int_0^1 \left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \circ \left(\frac{1}{s+1}+\frac{1}{s-1}\right) ds \\ &\quad \circ \left(\frac{1}{s+1}-\frac{1}{s-1}\right) ds \circ \frac{ds}{s} \circ \frac{ds}{s} \end{aligned}$$

$$
\begin{aligned}
&-\frac{12}{\pi^2}\int_0^1\left(\frac{1}{s+1}+\frac{1}{s-1}\right)ds\circ\left(\frac{1}{s+1}-\frac{1}{s-1}\right)ds\\
&\quad\circ\left(\frac{1}{s+1}+\frac{1}{s-1}\right)ds\circ\frac{ds}{s}\circ\frac{ds}{s}\\
&-\frac{12}{\pi^2}\int_0^1\left(\frac{1}{s+1}-\frac{1}{s-1}\right)ds\circ\left(\frac{1}{s+1}+\frac{1}{s-1}\right)ds\\
&\quad\circ\left(\frac{1}{s+1}+\frac{1}{s-1}\right)ds\circ\frac{ds}{s}\circ\frac{ds}{s}\\
&-\frac{6}{\pi^2}\int_0^1\left(\frac{1}{s+1}+\frac{1}{s-1}\right)ds\circ\frac{ds}{s}\circ\left(\frac{1}{s+1}-\frac{1}{s-1}\right)ds\circ\frac{ds}{s}\circ\frac{ds}{s}\\
&-\frac{6}{\pi^2}\int_0^1\left(\frac{1}{s+1}+\frac{1}{s-1}\right)ds\circ\left(\frac{1}{s+1}-\frac{1}{s-1}\right)ds\circ\frac{ds}{s}\circ\frac{ds}{s}\circ\frac{ds}{s}\\
&-\frac{6}{\pi^2}\int_0^1\left(\frac{1}{s+1}-\frac{1}{s-1}\right)ds\circ\left(\frac{1}{s+1}+\frac{1}{s-1}\right)ds\circ\frac{ds}{s}\circ\frac{ds}{s}\circ\frac{ds}{s}\\
&+\frac{12}{\pi^2}\int_0^1\left(\frac{1}{s+1}-\frac{1}{s-1}\right)ds\circ\frac{ds}{s}\circ\frac{ds}{s}\circ\frac{ds}{s}\circ\frac{ds}{s}.
\end{aligned}
$$

Once again we write everything in terms of hyperlogarithms

$$
\begin{aligned}
\mathrm{m}_3(\tilde{P}) = &-\frac{12}{\pi^2}(I_{1,2,2}(-1:-1:-1:1)-I_{1,2,2}(-1:-1:1:1)+I_{1,2,2}(1:-1:-1:1)\\
&-I_{1,2,2}(1:-1:1:1)+I_{1,2,2}(-1:1:-1:1)-I_{1,2,2}(-1:1:1:1)\\
&+I_{1,2,2}(1:1:-1:1)-I_{1,2,2}(1:1:1:1))\\
&-\frac{12}{\pi^2}(3I_{1,1,3}(-1:-1:-1:1)-3I_{1,1,3}(1:1:1:1)+I_{1,1,3}(-1:-1:1:1)\\
&-I_{1,1,3}(1:-1:1:1)+I_{1,1,3}(1:-1:-1:1)-I_{1,1,3}(-1:1:1:1)\\
&+I_{1,1,3}(-1:1:-1:1)-I_{1,1,3}(1:1:-1:1))\\
&-\frac{6}{\pi^2}(I_{2,3}(-1:-1:1)-I_{2,3}(-1:1:1)+I_{2,3}(1:-1:1)-I_{2,3}(1:1:1))\\
&-\frac{6}{\pi^2}(2I_{1,4}(-1:-1:1)-2I_{1,4}(1:1:1))+\frac{12}{\pi^2}(I_5(-1:1)-I_5(1:1)).
\end{aligned}
$$

We now write everything in terms of polylogarithms.

$$
\begin{aligned}
\mathrm{m}_3(\tilde{P}) = &\frac{12}{\pi^2}(\mathrm{Li}_{1,2,2}(1,1,-1)-\mathrm{Li}_{1,2,2}(1,-1,1)+\mathrm{Li}_{1,2,2}(-1,1,-1)-\mathrm{Li}_{1,2,2}(-1,-1,1)\\
&+\mathrm{Li}_{1,2,2}(-1,-1,-1)-\mathrm{Li}_{1,2,2}(-1,1,1)+\mathrm{Li}_{1,2,2}(1,-1,-1)-\mathrm{Li}_{1,2,2}(1,1,1))\\
&+\frac{12}{\pi^2}(3\mathrm{Li}_{1,1,3}(1,1,-1)-3\mathrm{Li}_{1,1,3}(1,1,1)+\mathrm{Li}_{1,1,3}(1,-1,1)-\mathrm{Li}_{1,1,3}(-1,-1,1)\\
&+\mathrm{Li}_{1,1,3}(-1,1,-1)-\mathrm{Li}_{1,1,3}(-1,1,1)+\mathrm{Li}_{1,1,3}(-1,-1,-1)-\mathrm{Li}_{1,1,3}(1,-1,-1))\\
&-\frac{6}{\pi^2}(\mathrm{Li}_{2,3}(1,-1)-\mathrm{Li}_{2,3}(-1,1)+\mathrm{Li}_{2,3}(-1,-1)-\mathrm{Li}_{2,3}(1,1))\\
&-\frac{6}{\pi^2}(2\mathrm{Li}_{1,4}(1,-1)-2\mathrm{Li}_{1,4}(1,1))+\frac{12}{\pi^2}(\mathrm{Li}_5(1)-\mathrm{Li}_5(-1)).
\end{aligned}
$$

We use the multiple zeta value data mine [3] in order to simplify the multiple polylogarithms of length 3 in the above expression. We have

$$\begin{aligned}
&\mathrm{Li}_{1,2,2}(1,1,-1)-\mathrm{Li}_{1,2,2}(1,-1,1)+\mathrm{Li}_{1,2,2}(-1,1,-1)-\mathrm{Li}_{1,2,2}(-1,-1,1)\\
&\quad+\mathrm{Li}_{1,2,2}(-1,-1,-1)-\mathrm{Li}_{1,2,2}(-1,1,1)+\mathrm{Li}_{1,2,2}(1,-1,-1)-\mathrm{Li}_{1,2,2}(1,1,1)\\
&=\frac{171}{32}\zeta(5)-\frac{23\pi^2}{48}\zeta(3)-8\mathrm{Li}_{1,1,3}(1,1,-1),
\end{aligned}$$

and

$$\begin{aligned}
&3\mathrm{Li}_{1,1,3}(1,1,-1)-3\mathrm{Li}_{1,1,3}(1,1,1)+\mathrm{Li}_{1,1,3}(1,-1,1)-\mathrm{Li}_{1,1,3}(-1,-1,1)\\
&\quad+\mathrm{Li}_{1,1,3}(-1,1,-1)-\mathrm{Li}_{1,1,3}(-1,1,1)+\mathrm{Li}_{1,1,3}(-1,-1,-1)-\mathrm{Li}_{1,1,3}(1,-1,-1)\\
&=-\frac{97}{16}\zeta(5)+\frac{25\pi^2}{48}\zeta(3)+4\mathrm{Li}_{1,1,3}(1,1,-1).
\end{aligned}$$

Theorem 2 yields

$$\mathrm{Li}_{2,3}(1,-1)-\mathrm{Li}_{2,3}(-1,1)+\mathrm{Li}_{2,3}(-1,-1)-\mathrm{Li}_{2,3}(1,1)=\frac{93}{8}\zeta(5)-\frac{49\pi^2}{48}\zeta(3),$$

and

$$2\mathrm{Li}_{1,4}(1,-1)-2\mathrm{Li}_{1,4}(1,1)=-\frac{93}{16}\zeta(5)+\frac{\pi^2}{2}\zeta(3).$$

In addition, we have

$$\mathrm{Li}_5(1)-\mathrm{Li}_5(-1)=\frac{31}{16}\zeta(5).$$

Putting everything together, we obtain,

$$\mathrm{m}_3(\tilde{P})=-\frac{81}{4\pi^2}\zeta(5)+\frac{29}{8}\zeta(3)-\frac{48}{\pi^2}\mathrm{Li}_{1,1,3}(1,1,-1). \tag{11}$$

As before, we may seek to write this formula in terms of length one polylogarithms. Equation (64) from [19] says

$$\begin{aligned}
\int_0^1 \log(x)\log^2(1-x)\log(1+x)\frac{dx}{x}&=4\mathrm{Li}_5\left(\frac{1}{2}\right)-\frac{7}{2}\zeta(5)+4\log 2\mathrm{Li}_4\left(\frac{1}{2}\right)\\
&\quad-\frac{\pi^2}{16}\zeta(3)+\frac{7\log^2 2}{4}\zeta(3)-\frac{\log^3 2}{9}\pi^2+\frac{2\log^5 2}{15}.
\end{aligned} \tag{12}$$

After some manipulations, we can see that the above integral equals

$$2\mathrm{Li}_{1,1,3}(-1,1,1)+2\mathrm{Li}_{1,1,3}(-1,-1,1)+2\mathrm{Li}_{1,1,3}(1,-1,-1). \tag{13}$$

On the other hand, we use the multiple zeta value data mine [3] again to obtain

$$\begin{aligned}
&\mathrm{Li}_{1,1,3}(-1,1,1)+\mathrm{Li}_{1,1,3}(-1,-1,1)+\mathrm{Li}_{1,1,3}(1,-1,-1)\\
&=-\mathrm{Li}_{1,1,3}(1,1,-1)-\frac{23}{32}\zeta(5)+\frac{5\pi^2}{96}\zeta(3).
\end{aligned} \tag{14}$$

Combining (12), (13), and (14), we get

$$\begin{aligned}
\mathrm{Li}_{1,1,3}(1,1,-1)&=-2\mathrm{Li}_5\left(\frac{1}{2}\right)+\frac{33}{32}\zeta(5)-2\log 2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{\pi^2}{12}\zeta(3)\\
&\quad-\frac{7\log^2 2}{8}\zeta(3)+\frac{\log^3 2}{18}\pi^2-\frac{\log^5 2}{15}.
\end{aligned}$$

Combining the above with (11), we finally obtain

$$\mathrm{m}_3(\tilde{P}) = \frac{96}{\pi^2}\mathrm{Li}_5\left(\frac{1}{2}\right) - \frac{279}{4\pi^2}\zeta(5) + \frac{96\log 2}{\pi^2}\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{3}{8}\zeta(3) + \frac{42\log^2 2}{\pi^2}\zeta(3) - \frac{8\log^3 2}{3} + \frac{16\log^5 2}{5\pi^2}.$$

□

6 Concluding remarks

We have introduced a new method for constructing Mahler measure examples from known formulas involving polylogarithms. In principle the possibilities of this method are endless. For example, it can be combined with Akatsuka's Theorem 4 in order to recover a formula for $\mathrm{m}_k\left(1+\left(\frac{1+y}{1+z}\right)x\right)$ not only for $k \leq 3$ as we have done in this note, but also for general k. The general formula is quite complicated and we do not have a systematic way to reduce polylogarithms evaluated at ± 1 of arbitraty length to polylogarithms of length one. Therefore we decided to spare the reader the details. It would be interesting to find a simpler method of proving formulas for $\mathrm{m}_k\left(1+\left(\frac{1+y}{1+z}\right)x\right)$ that would lead directly to length-one polylogarithms.

The difference between [11] and [12] is that in the first work we make the substitution $a \to a\frac{1-z}{1+z}$ while in the second work we make $a \to \left(\frac{1-z_1}{1+z_1}\right)\cdots\left(\frac{1-z_n}{1+z_n}\right)$ all together. The current paper is analogous to [11] in the sense that we make the substitution $a \to \frac{1+y}{1+z}$ with just one step. It is not hard to consider the analogous processus to the one of [12] for the current work. Theorem 1, when iterated, yields

$$\mathrm{m}(\tilde{P}) = \left(\frac{2}{\pi^2}\right)^n \int_{\substack{0\leq s_1\leq s_2\leq 1\\ \cdots \\ 0\leq s_{2n-1}\leq s_{2n}\leq 1}} \sum_{(\sigma_1,\ldots,\sigma_n)\in\{-1,1\}^n} \mathrm{m}_k\left(P_{s_2^{\sigma_1}\cdots s_{2n}^{\sigma_n}}\right) \times \left(\frac{1}{s_1+1}-\frac{1}{s_1-1}\right)ds_1\frac{ds_2}{s_2}\cdots\left(\frac{1}{s_{2n-1}+1}-\frac{1}{s_{2n-1}-1}\right)ds_{2n-1}\frac{ds_{2n}}{s_{2n}}, \quad (15)$$

where

$$\tilde{P} = P_{\left(\frac{1+y_1}{1+z_1}\right)\cdots\left(\frac{1+y_n}{1+z_n}\right)}.$$

Formula (15) can be applied to prove Theorem 3 directly, without passing through (5). We have not been able to apply (15) to any other case because we always have some polylogarithms of weight higher than one evaluated in a product of variables, and polylogarithms of higher weight are not multiplicative (they satisfy more complicated functional equations).

Finally, we mention the possibility of combining the method of this paper with the one described in [12,14]. This has the potential of providing formulas such as the one in Theorem 3. When we try to apply this approach for new formulas, we seem to find again the same difficulties regarding polylogarithms evaluated in product of variables as described above for the application of (15). It would be very interesting to find a way to overcome these issues.

Acknowledgements
We would like to thank the anonymous referee for several comments that helped improve the exposition of the text. This work was supported by he Natural Sciences and Engineering Research Council of Canada [Discovery Grant 355412-2013].

Received: 7 March 2016 Accepted: 21 June 2016
Published online: 01 September 2016

References

1. Akatsuka, H.: Zeta Mahler measures. J. Number. Theor. **129**(11), 2713–2734 (2009)
2. Bigotte, M., Jacob, G., Oussous, N.E., Petitot, M.: Lyndon words and shuffle algebras for generating the colored multiple zeta values relations tables. Theor. Comput. Sci. **273**(1–2), 271–282 (2002)
3. Blümlein, J., Broadhurst, D.J., Vermaseren, J.A.M.: The multiple zeta value data mine. Comput. Phys. Comm. **181**(3), 582–625 (2010)
4. Borwein, J.M., Bradley, D.M., Broadhurst, D.J.: Evaluations of k-fold Euler/Zagier sums: a compendium of results for arbitrary k. Electronic. J. Combin. **4**(2), 5 (1997)
5. Borwein, D., Borwein, J.M., Girgensohn, R.: Explicit evaluation of Euler sums. Proc. Edinburgh. Math. Soc. **38**(2), 277–294 (1995)
6. Boyd, D.W.: Speculations concerning the range of Mahler's measure. Canad. Math. Bull. **24**, 453–469 (1981)
7. Boyd, D.W.: Mahler's measure and special values of L-functions. Experiment. Math. **7**(1), 37–82 (1998)
8. Deninger, C.: Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbb{Z}^n$-actions. J. Am. Math. Soc. **10**(2), 259–281 (1997)
9. Goncharov AB. Polylogarithms in arithmetic and geometry. In: Proceedings of the International Congress of Mathematicians, vol. 1. Zürich (1994)
10. Kurokawa, N., Lalín, M., Ochiai, H.: Higher Mahler measures and zeta functions. Acta. Arith. **135**(3), 269–297 (2008)
11. Lalín, M.N.: Some examples of Mahler measures as multiple polylogarithms. J. Number. Theor. **103**(1), 85–108 (2003)
12. Lalín, M.N.: Mahler measure of some n-variable polynomial families. J. Number. Theor. **116**(1), 102–139 (2006)
13. Lalin MN. Higher Mahler measure as a Massey product in Deligne cohomology. Low-dimensional topology and number theory. In: Abstracts from the workshop held August 15–21, 2010. Organized by Paul E. Gunnells, Walter Neumann, Adam S. Sikora and Don Zagier. Oberwolfach Reports, vol. 7, pp. 2101–2163. (2010)
14. Lalín, M.N., Lechasseur, J.S.: Higher Mahler measure of an n-variable family. Acta. Arith. (2016, **in press**)
15. Lehmer, D.H.: Factorization of certain cyclotomic functions. Ann. Math. **2**(34), 461–479 (1933)
16. Mahler, K.: On some inequalities for polynomials in several variables. J. London. Math. Soc. **37**, 341–344 (1962)
17. Rodriguez-Villegas, F: Modular Mahler measures. I. In: Topics in number theory (University Park, PA, 1997), pp 17–48. Math. Appl., 467, Kluwer Acad. Publ., Dordrecht (1999)
18. Smyth, C.J.: An explicit formula for the Mahler measure of a family of 3-variable polynomials. J. Th. Nombres. Bordeaux. **14**, 683–700 (2002)
19. Vermaseren, J.A.M.: Harmonic sums, Mellin transforms and integrals. Int. J. Modern Phys. A. **14**, 2037–2076 (1999)